Chapter 8 – Number and Proof

Solutions to Exercise 8A

1 **a** As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= 2p + 2q \\ &= 2(p + q), \end{aligned}$$

is an even number.

b As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (2p)(2q) \\ &= 4pq \\ &= 2(2pq), \end{aligned}$$

is an even number.

2 As m and n are odd, $m = 2p + 1$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= (2p + 1) + (2q + 1) \\ &= 2p + 2q + 2 \\ &= 2(p + q + 1), \end{aligned}$$

is an even number.

3 As m is even and n is odd, $m = 2p$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= 2p(2q + 1) \\ &= 2(2pq + p), \end{aligned}$$

is an even number.

4 **a** If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3p)(7q) \\ &= 21pq, \end{aligned}$$

is divisible by 21.

b If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m^2n &= (3p)^2(7q) \\ &= 9p^2(7q) \\ &= 63p^2q \end{aligned}$$

is divisible by 63.

5 If m and n are perfect squares then $m = a^2$ and $n = b^2$ for some $a, b \in \mathbb{Z}$. Therefore,

$$mn = (a^2)(b^2) = (ab)^2,$$

is also a perfect square.

6 Expanding both brackets gives,

$$\begin{aligned} &(m + n)^2 - (m - n)^2 \\ =&m^2 + 2mn + n^2 - (m^2 - 2mn + n^2) \\ =&m^2 + 2mn + n^2 - m^2 + 2mn - n^2 \\ =&4mn, \end{aligned}$$

which is divisible by 4.

7 (Method 1) If n is even then n^2 is even and $6n$ is even. Therefore the expression is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

(Method 2) If n is even then $n = 2k$

where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 4k^2 - 12k + 4 + 1 \\ &= 2(2k^2 - 6k + 2) + 1, \quad = \end{aligned}$$

is odd.

8 (Method 1) If n is odd then n^2 is odd and $8n$ is even. Therefore the expression is of the form

odd + even + odd = even.

(Method 2) If n is odd then $n = 2k + 1$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 + 8n + 5 &= (2k + 1)^2 + 8(2k + 1) + 3 \\ &= 4k^2 + 4k + 1 + 16k + 8 + 3 \\ &= 4k^2 + 20k + 12 \\ &= 2(2k^2 + 10k + 6), \end{aligned}$$

is even.

9 First suppose n is even. Then $5n^2$ and $3n$ are both even. Therefore the expression is of the form

even + even + odd = odd.

Now suppose n is odd. Then $5n^2$ and $3n$ are both odd. Therefore the expression is of the form

odd + odd + odd = odd.

10 Firstly, if $x > y$ then $x - y > 0$. Secondly, since x and y are positive, $x + y > 0$.

Therefore,

$$\begin{aligned} & x^4 - y^4 \\ =& (x^2 - y^2)(x^2 + y^2) \\ =& (x - y)(x + y)(x^2 + y^2) \\ =& \overbrace{(x - y)}^{\text{positive}}\overbrace{(x + y)}^{\text{positive}}\overbrace{(x^2 + y^2)}^{\text{positive}} \\ >& 0. \end{aligned}$$

Therefore, $x^4 > y^4$.

11 We have,

$$\begin{aligned} & x^2 + y^2 - 2xy \\ =& x^2 - 2xy + y^2 \\ =& (x - y)^2 \\ \geq& 2xy. \end{aligned}$$

Therefore, $x^2 + y^2 \geq 2xy$.

12 **a** We prove that Alice is a knave, and Bob is a knight.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice and Bob are both knaves
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are not both knaves
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

b We prove that Alice is a knave, and Bob is a knight.

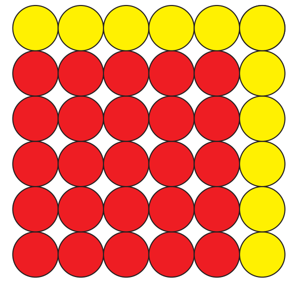
Suspect Alice is a knight

$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ They are both of the same kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is lying
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is a knight and a knave.
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are of a different kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

c We will prove that Alice is a knight, and Bob is a knave.

Suppose Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is telling the truth
$\Rightarrow$ Neither of them are knaves
$\Rightarrow$ Both of them are knights
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is lying
$\Rightarrow$ At least one of them is a knave
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Alice is a knight, and Bob is a knave.

13 **a** In the diagram below, there are 11 yellow tiles. We can also count the yellow tiles by subtracting the number of red tiles, 5^2, from the total number of tiles, 6^2. Therefore $11 = 6^2 - 5^2$.



b Every odd number is of the form $2k + 1$ for some $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned}(k+1)^2 - k^2 &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1,\end{aligned}$$

so that every odd number can be written as the difference of two squares.

c Since $101 = 2 \times 50 + 1$, we have,

$$51^2 - 50^2 = 101.$$

14 **a** Since

$$\frac{9}{10} = \frac{99}{110} \text{ and } \frac{10}{11} = \frac{100}{110},$$

it is clear that

$$\frac{10}{11} > \frac{9}{10}.$$

b We have,

$$\begin{aligned}&\frac{n}{n+1} - \frac{n-1}{n} \\ &= \frac{n^2}{n(n+1)} - \frac{n(n-1)}{n(n+1)} \\ &= \frac{n^2 - n(n-1)}{n(n+1)} \\ &= \frac{n^2 - n^2 + n}{n(n+1)} \\ &= \frac{1}{n(n+1)} \\ &> 0\end{aligned}$$

since $n(n+1) > 0$. Therefore,

$$\frac{n}{n+1} > \frac{n-1}{n}.$$

15 a We have,

$$\begin{aligned}&\frac{1}{10}-\frac{1}{11}\\=&\frac{11}{110}-\frac{10}{110}\\=&\frac{1}{110}\\<&\frac{1}{100},\end{aligned}$$

since 110 > 100.

b We have,

$$\begin{aligned}\frac{1}{n}-\frac{1}{n+1}&=\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}\\&=\frac{n+1-n}{n(n+1)}\\&=\frac{1}{n(n+1)},\\&=\frac{1}{n^2+n},\\&<\frac{1}{n^2},\end{aligned}$$

since $n^2+n>n^2$.

16 We have,

$$\begin{aligned}&\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\\=&\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}\\=&\frac{2a^2+2b^2}{4}-\frac{a^2+2ab+b^2}{4}\\=&\frac{2a^2+2b^2-a^2-2ab-b^2}{4}\\=&\frac{a^2-2ab+b^2}{4}\\=&\frac{(a-b)^2}{4}\\\geq&0.\end{aligned}$$

17 a Expanding gives,

$$\begin{aligned}&(x-y)(x^2+xy+y^2)\\=&x^3+x^2y+xy^2-x^2y-xy^2-y^3\\=&x^3-y^3,\end{aligned}$$

which is the difference of two cubes.

b Completing the square by treating y as a constant gives,

$$\begin{aligned}&x^2+yx+y^2\\=&x^2+yx+\frac{y^2}{4}-\frac{y^2}{4}+y^2\\=&\left(x^2+yx+\frac{y^2}{4}\right)+\frac{3y^2}{4}\\=&\left(x+\frac{y}{2}\right)^2+\frac{3y^2}{4}\\\geq&0\end{aligned}$$

c Firstly, if $x\geq y$ then $x-y\geq 0$. Therefore,

$$\begin{aligned}&x^3-y^3\\=&\overbrace{(x-y)}^{\geq 0}\overbrace{(x^2+xy+y^2)}^{\geq 0}\\\geq&0.\end{aligned}$$

Therefore, $x^3>y^3$.

18 a Let D be the distance to and from work. The time taken to get to work is $D/12$ and the time taken to get home from work is $D/24$. The total distance is $2D$ and the total time is

$$\frac{D}{12} + \frac{D}{24}$$
$$= \frac{2D}{24} + \frac{D}{24}$$
$$= \frac{3D}{24}$$
$$= \frac{D}{8}$$

The average speed will then be

$$\text{distance} \div \text{time}$$
$$= 2D \div \frac{D}{8}$$
$$= 2D \times \frac{8}{D}$$
$$= 16 \text{ km/hour.}$$

b Let D be the distance to and from work. The time taken to get to work is D/a and the time taken to get home from work is D/b. The total distance is $2D$ and the total time is

$$\frac{D}{a} + \frac{D}{b}$$
$$= \frac{bD}{ab} + \frac{aD}{ab}$$
$$= \frac{aD + bD}{ab}$$
$$= \frac{(a+b)D}{ab}$$

The average speed will then be

$$\text{distance} \div \text{time}$$
$$= 2D \div \frac{(a+b)D}{ab}$$
$$= 2D \times \frac{ab}{(a+b)D}$$
$$= \frac{2ab}{a+b} \text{ km/hour.}$$

c We first note that $a + b > 0$. Secondly,

$$\frac{a+b}{2} - \frac{2ab}{a+b}$$
$$= \frac{(a+b)^2}{2(a+b)} - \frac{4ab}{2(a+b)}$$
$$= \frac{(a+b)^2 - 4ab}{2(a+b)}$$
$$= \frac{a^2 + 2ab + b^2 - 4ab}{2(a+b)}$$
$$= \frac{a^2 - 2ab + b^2}{2(a+b)}$$
$$= \frac{(a-b)^2}{2(a+b)}$$
$$\geq 0$$

since $(a - b) \geq 0$ and $a + b > 0$. Therefore,

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}.$$

Solutions to Exercise 8B

1 **a** $P : 1 > 0$ (true)
not $P : 1 \leq 0$ (false)

b P : 4 is divisible by 8 (false)
not P : 4 is not divisible by 8 (true)

c P : Each pair of primes has an even sum (false)
not P : Some pair of primes does not have an even sum (true)

d P : Some rectangle has 4 sides of equal length (true)
not P : No rectangle has 4 sides of equal length (false)

2 **a** P : 14 is divisible by 7 and 2 (true)
not P : 14 is not divisible by 7 or 14 is not divisible by 2 (false)

b P : 12 is divisible by 3 or 4 (true)
not P : 12 is not divisible by 4 and 12 is not divisible by 3 (false)

c P : 15 is divisible by 3 and 6 (false)
not P : 15 is not divisible by 3 or 15 is not divisible by 6 (true)

d P : 10 is divisible by 2 or 5 (true)
not P : 10 is not divisible by 2 or 10 is not divisible by 5 (false)

3 We will prove that Alice is a knave, and Bob is a knave.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice is a knight OR Bob is a knave
$\Rightarrow$ Bob is a knave, as Alice is not a knight
$\Rightarrow$ Alice and Bob are both knaves.

4 **a** If there are no clouds in the sky, then it is not raining.

b If you are not happy, then you are not smiling.

c If $2x \neq 2$, then $x \neq 1$.

d If $x^5 \leq y^5$, then $x \leq y$.

e Option 1: If n is not odd, then n^2 is not odd.
Option 2: If n is even, then n^2 is even.

f Option 1: If mn is not odd, then n is not odd or m is not odd.
Option 2: If If mn is even, then n is even or m is even.

g Option 1: If n and n are not both even or both odd, then $m + n$ is not even.
Option 2: If n and n are not both even or both odd, then $m + n$ is odd.

5 **a** Contrapositive: If n is even then $3n + 5$ is odd.
Proof: Suppose n is even. Then

$n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n + 5 &= 3(2k) + 5 \\ &= 6k + 5 \\ &= 6k + 4 + 1 \\ &= 2(3k + 2) + 1 \end{aligned}$$

is odd.

b Contrapositive: If n is even, then n^2 is even.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

is even.

c Contrapositive: If n is even, then $n^2 - 8n + 3$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 - 8n + 3 &= (2k)^2 - 8(2k) + 3 \\ &= 4k^2 - 16k + 3 \\ &= 4k^2 - 16k + 2 + 1 \\ &= 2(2k^2 - 8k + 1) + 1 \end{aligned}$$

is odd.

d Contrapositive: If n is divisible by 3, then n^2 is divisible by 3.
Proof: Suppose n is divisible by 3. Then $n = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2) \end{aligned}$$

is divisible by 3.

e Contrapositive: If n is even, then $n^3 + 1$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 + 1 &= (2k)^3 + 1 \\ &= 8k^3 + 1 \\ &= 2(4k^3) + 1 \end{aligned}$$

is odd.

f Contrapositive: If m or n are divisible by 3, then mn is divisible by 3.
Proof: If m or n is divisible by 3 then we can assume that m is divisible by 3. Then, $m = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3k)n \\ &= 3(kn) \end{aligned}$$

is divisible by 3.

g Contrapositive: If $m = n$, then $m + n$ is even.
Proof: Suppose that $m = n$. Then

$$\begin{aligned} m + n &= n + n \\ &= 2n \end{aligned}$$

is even.

6 **a** Contrapositive: If $x \geq 0$, then $x^2 + 3x \geq 0$.
Proof: Suppose that $x \geq 0$. Then,

$$x^2 + 3x = x(x + 3) \geq 0,$$

since $x \geq 0$ and $x + 3 \geq 0$.

b Contrapositive: If $x \leq -1$, then $x^3 - x \leq 0$.
Proof: Suppose that $x \leq -1$. Then,

$$x^3 - x = x^2(x - 1) \leq 0,$$

since $x^2 \geq 0$ and $x - 1 \geq 0$.

c Contrapositive: If $x < 1$ and $y < 1$, then $x + y < 2$.
Proof: If $x < 1$ and $y < 1$ then,

$$x + y < 1 + 1 = 2,$$

as required.

d Contrapositive: If $x < 3$ and $y < 2$, then $2x + 3y < 12$.
Proof: If $x < 3$ and $y < 2$ then,

$$2x + 3y < 2 \times 3 + 3 \times 2 = 6 + 6 = 12,$$

as required.

7 a Contrapositive: If m is odd or n is odd, then mn is odd or $m + n$ is odd.

b Proof:
(Case 1) Suppose m is odd and n is odd. Then clearly mn is odd.
(Case 2) Suppose m is odd and n is even. Then clearly $m + n$ will be odd.
It is likewise, if m is even and n is odd.

8 a We rationalise the right hand side to give,

$$\begin{aligned}
&\frac{x-y}{\sqrt{x}+\sqrt{y}}\\
&=\frac{x-y}{\sqrt{x}+\sqrt{y}}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}\\
&=\frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)}\\
&=\frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{(x-y)}\\
&=\sqrt{x}-\sqrt{y}.
\end{aligned}$$

b If $x > y$ then $x - y > 0$. Then, using the above equality, we see that,

$$\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}}>0,$$

since the numerator and denominator are both positive. Therefore, $\sqrt{x} > \sqrt{y}$.

c Contrapositive: If $\sqrt{x} \le \sqrt{y}$, then $x \le y$.
Proof: If $\sqrt{x} \le \sqrt{y}$ then, since both sides are positive, we can square both sides to give $x \le y$.

Solutions to Exercise 8C

1 If all three angles are less than 60°, then the sum of interior angles of the triangle would be less than 180°. This is a contradiction as the sum of interior angles is exactly 180°.

2 Suppose there is some least positive rational number $\frac{p}{q}$. Then since,

$$\frac{p}{2q} < \frac{p}{q},$$

there is some lesser positive rational number, which is a contradiction. Therefore, there is no least positive rational number.

3 Suppose that $\sqrt{p}$ is an integer. Then

$$\sqrt{p} = n,$$

for some $n \in \mathbb{Z}$. Squaring both sides gives

$$p = n^2.$$

Since $n \neq 1$, this means that p has three factors: $1, n$ and n^2. This is a contradiction since every prime number has exactly two factors.

4 Suppose that x is rational so that $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 3^x = 2 \\ \Rightarrow\quad & 3^{\frac{p}{q}} = 2 \\ \Rightarrow\quad & \left(3^{\frac{p}{q}}\right)^q = 2^q \\ \Rightarrow\quad & 3^p = 2^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

5 Suppose that $\log_2 5$ is rational so that $\log_2 5 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 2^{\frac{p}{q}} = 5 \\ \Rightarrow\quad & 2^{\frac{p}{q}} = 5 \\ \Rightarrow\quad & \left(2^{\frac{p}{q}}\right)^q = 5^q \\ \Rightarrow\quad & 2^p = 5^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

6 Suppose the contrary, so that $\sqrt{x}$ is rational. Then

$$\sqrt{x} = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$. Then, squaring both sides of the equation gives,

$$x = \frac{p^2}{q^2},$$

where $p^2, q^2 \in \mathbb{Z}$. Therefore, x is rational, which is a contradiction.

7 Suppose, on the contrary that $a + b$ is rational. Then

$$b = \overbrace{(a+b)}^{\text{rational}} - \overbrace{b}^{\text{rational}}.$$

Therefore, b is the difference of two rational numbers, which is rational. This is a contradiction.

8 Suppose b and c are both natural numbers. Then

$$c^2 - b^2 = 4$$
$$(c - b)(c + b) = 4.$$

The only factors of 4 are 1, 2 and 4. And since $c + b > c - b$,

$$c - b = 1 \text{ and } c + b = 4.$$

Adding these two equations gives $2c = 5$ so that $c = \frac{2}{5}$, which is not a whole number.

9 Suppose that there are two different solutions, x_1 and x_2. Then,

$$ax_1 + b = c \text{ and } ax_2 + b = c.$$

Equating these two equations gives,

$$ax_1 + b = ax_2 + b$$
$$ax_1 = ax_2$$
$$x_1 = x_2, \qquad (\text{since } a \neq 0)$$

which is a contradiction since the two solutions were assumed to be different.

10 **a** Every prime $p > 2$ is odd since if it were even then p would be divisible by 2.

b Suppose there are two primes p and p such that $p + q = 1001$. Then since the sum of two odd numbers is even, one of the primes must be 2. Assume $p = 2$ so that $q = 999$. Since 999 is not prime, this gives a contradiction.

11 **a** Suppose that

$$42a + 7b = 1.$$

Then

$$7(6a + b) = 1.$$

This implies that 1 is divisible by 7, which is a contradiction since the only factor of 1 is 1.

b Suppose that

$$15a + 21b = 2.$$

Then

$$3(5a + 7b) = 2.$$

This implies that 2 is divisible by 3, which is a contradiction since the only factors of 2 are 1 and 2.

12 **a** Contrapositive: If n is not divisible by 3, then n^2 is not divisible by 3.
Proof: If n is not divisible by 3 then either $n = 3k + 1$ or $n = 3k + 2$.
(Case 1) If $n = 3k + 1$ then,

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

is not divisible by 3.
(Case 2) If $n = 3k + 2$ then,

$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

is not divisible by 3.

b This will be a proof by contradiction. Suppose $\sqrt{3}$ is rational so that $\sqrt{3} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^2 = 3q^2 \qquad (1)$$
$$\Rightarrow \quad p^2 \text{ is divisible by 3}$$
$$\Rightarrow \quad p \text{ is divisible by 3}$$
$$\Rightarrow \quad p = 3k \text{ for some } k \in \mathbb{N}$$
$$\Rightarrow \quad (3k)^2 = 3q^2 \text{(substitituting into (1))}$$
$$\Rightarrow \quad 3q^2 = 9k^2$$
$$\Rightarrow \quad q^2 = 3k^2$$
$$\Rightarrow \quad q^2 \text{ is divisible by 3}$$
$$\Rightarrow \quad q \text{ is divisible by 3.}$$

So p and q are both divisible by 3, which contradicts the fact that they have no factors in common.

13 a Contrapositive: If n is odd, then n^3 is odd.
Proof: If n is odd then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k+1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^3 + 6k^2 + 3k) + 1$$

is odd. Otherwise, we can simply quote the fact that the product of 3 odd numbers will be odd.

b This will be a proof by contradiction. Suppose $\sqrt[3]{2}$ is rational so that $\sqrt[3]{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^3 = 2q^3 \qquad (1)$$
$$\Rightarrow \quad p^3 \text{ is divisible by 2}$$
$$\Rightarrow \quad p \text{ is divisible by 2}$$
$$\Rightarrow \quad p = 2k \text{ for some } k \in \mathbb{N}$$
$$\Rightarrow \quad (2k)^3 = 2q^3 \text{(substitituting into (1))}$$
$$\Rightarrow \quad 2q^3 = 8k^3$$
$$\Rightarrow \quad q^3 = 4k^3$$
$$\Rightarrow \quad q^3 \text{ is divisible by 2}$$
$$\Rightarrow \quad q \text{ is divisible by 2.}$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

14 This will be a proof by contradiction, so we suppose there is some $a, b \in \mathbb{Z}$ such that

$$a^2 - 4b - 2 = 0$$
$$\Rightarrow a^2 = 4b + 2$$
$$\Rightarrow a^2 = 2(2b+1) \qquad (1)$$

which means that a^2 is even. However, this implies that a is even, so that $a = 2k$, for some $k \in \mathbb{Z}$. Substituting this into equation (1) gives,

$$(2k)^2 = 2(2b+1)$$
$$4k^2 = 2(2b+1)$$
$$2k^2 = 2b + 1$$
$$2k^2 - 2b = 1$$
$$2(k^2 - b) = 1.$$

This implies that 1 is divisible by 2, which is a contradiction since the only factor of 1 is 1.

15 a Suppose on the contrary, that $a > \sqrt{n}$

and $b > \sqrt{n}$. Then

$$ab > \sqrt{n}\sqrt{n} = n,$$

which is a contradiction since $ab = n$.

b If 97 were not prime then we could write $97 = ab$ where $1 < a < b < n$. By the previous question, we know that

$$a \le \sqrt{97} < \sqrt{100} = 10.$$

Therefore a is one of

$$\{2, 3, 4, 5, 6, 7, 8, 9\}.$$

However 97 is not divisible by any of these numbers, which is a contradiction. Therefore, 97 is a prime number.

16 **a** Let $m = 4n + r$ where $r = 0, 1, 2, 3$.
($r = 0$) We have,

$$\begin{aligned} m^2 &= (4n)^2 \\ &= 16n^2 \\ &= 4(4n^2) \end{aligned}$$

is divisible by 4.
($r = 1$) We have,

$$\begin{aligned} m^2 &= (4n+1)^2 \\ &= 16n^2 + 8n + 1 \\ &= 4(4n^2 + 2n) + 1 \end{aligned}$$

has a remainder of 1.
($r = 2$) We have,

$$\begin{aligned} m^2 &= (4n+2)^2 \\ &= 16n^2 + 16n + 4 \\ &= 4(4n^2 + 4n + 1) \end{aligned}$$

is divisible by 4.

($r = 3$) We have,

$$\begin{aligned} m^2 &= (4n+3)^2 \\ &= 16n^2 + 24n + 9 \\ &= 16n^2 + 24n + 8 + 1 \\ &= 4(4n^2 + 6n + 2) + 1 \end{aligned}$$

has a remainder of 1.

Therefore, the square of every integer is divisible by 4 or leaves a remainder of 1.

b Suppose the contrary, so that both a and b are odd. Then $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 = 4m + 1 \\ &= 4(k^2 + m^2 + k + m) + 2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 4, which is a contradiction.

17 **a** Suppose by way of contradiction either $a \ne c$ or $b \ne d$. Then clearly both $a \ne c$ and $b \ne d$. Therefore,

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ (b-d)\sqrt{2} &= c - a \\ \sqrt{2} &= \frac{c-a}{b-d} \end{aligned}$$

Since $\frac{c-a}{b-d} \in \mathbb{Q}$, this contradicts the irrationality of $\sqrt{2}$.

b Squaring both sides gives,

$$3 + 2\sqrt{2} = \left(c + d\sqrt{2}\right)^2$$
$$3 + 2\sqrt{2} = c^2 + 2cd\sqrt{2} + 2d^2$$
$$3 + 2\sqrt{2} = c^2 + 2d^2 + 2cd\sqrt{2}$$

Therefore

$$c^2 + 2d^2 = 3 \qquad (1)$$
$$cd = 1 \qquad (2)$$

Since c and d are integers, this implies that $c = d = 1$.

18 There are many ways to prove this result. We will take the most elementary approach (but not the most elegant). Suppose that

$$ax^2 + bx + c = 0 \qquad (1)$$

has a rational solution, $x = \frac{p}{q}$. We can assume that p and q have no factors in common (or else we could cancel).

Equation (1) then becomes

$$ax^2 + bx + c = 0$$
$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$$
$$ap^2 + bpq + cq^2 = 0 \qquad (2)$$

Since p and q cannot both be even, we need only consider three cases.

(Case 1) If p is odd and q is odd then equation (2) is of the form

$$\text{odd} + \text{odd} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.

(Case 2) If p is odd and q is even then equation (2) is of the form

$$\text{odd} + \text{even} + \text{even} = \text{odd} = 0.$$

This is not possible since 0 is even.

(Case 3) If p is even and q is odd then equation (2) is of the form

$$\text{even} + \text{even} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.

Solutions to Exercise 8D

1 **a** Converse: If $x = 1$, then $2x + 3 = 5$.
Proof: If $x = 1$ then

$$2x + 3 = 2 \times 1 + 3 = 5.$$

b Converse: If $n - 3$ is even, then n is odd.
Proof: If $n - 3$ is even then $n - 3 = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$n = 2k + 3 = 2k + 2 + 1 = 2(k + 1) + 1$$

is odd.

c Converse: If m is odd, then $m^2 + 2m + 1$ is even.
Proof 1: If m is odd then the expression $m^2 + 2m + 1$ is of the form,

$$\text{odd} + \text{even} + \text{odd} = \text{even}.$$

Proof 2: If m is odd then $m = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} & m^2 + 2m + 1 \\ &= (2k + 1)^2 + 2(k + 1) + 1 \\ &= 4k^2 + 4k + 1 + 2k + 2 + 1 \\ &= 4k^2 + 6k + 3 \\ &= 4k^2 + 6k + 2 + 1 \\ &= 2(2k^2 + 3k + 1) + 1, \end{aligned}$$

is clearly odd.

d Converse: If n is divisible by 5, then n^2 is divisible by 5.
Proof: If n is divisible by 5 then $n = 5k$ for some $k \in \mathbb{Z}$. Therefore,

$$n^2 = (5k)^2 = 25k^2 = 5(5k^2),$$

which is divisible by 5.

2 **a** Converse: If mn is a multiple of 4, then m and n are even.

b This statement is not true. For instance, 4×1 is a multiple of 4, and yet 1 is clearly not even.

3 **a** These statements are not equivalent.
($P \Rightarrow Q$) If Vivian is in China then she is in Asia, since Asia is a country in China.
($Q \not\Rightarrow P$) If Vivian is in Asia, she is not necessarily in China. For example, she could be in Japan.

b These statements are equivalent.
($P \Rightarrow Q$) If $2x = 4$, then dividing both sides by 2 gives $x = 2$.
($Q \Rightarrow P$) If $x = 2$, then multiplying both sides by 2 gives $2x = 4$.

c These statements are not equivalent.
($P \Rightarrow Q$) If $x > 0$ and $y > 0$ then $xy > 0$ since the product of two positive numbers is positive.
($Q \not\Rightarrow P$) If $xy > 0$, then it may not be true that $x > 0$ and $y > 0$. For example, $(-1) \times (-1) > 0$, however $-1 < 0$.

d These statements are equivalent.
($P \Rightarrow Q$) If m or n are even then mn will be even.
($Q \Rightarrow P$) If mn is even then either m or n are even since otherwise the product of two odds numbers would give an odd number.

4 ($\Rightarrow$) If $n + 1$ is odd then, $n + 1 = 2k + 1$, where $k \in \mathbb{Z}$. Therefore,

$$n + 2 = 2k + 2$$
$$= 2(k + 1),$$

so that $n + 2$ is even.
($\Leftarrow$) If $n + 2$ is even then, $n + 2 = 2k$, where $k \in \mathbb{Z}$. Therefore,

$$n + 1 = 2k - 1$$
$$= 2k - 2 + 1$$
$$= 2(k - 1) + 1$$

so that $n + 1$ is odd.

5 ($\Rightarrow$) Suppose that $n^2 - 4$ is prime. Since

$$n^2 - 4 = (n - 2)(n + 2)$$

expresses $n^2 - 4$ as the product of two numbers, either $n - 2 = 1$ or $n + 2 = 1$. Therefore, $n = 3$ or $n = -1$. However, n must be positive, so $n = 3$.
($\Leftarrow$) If $n = 3$ then

$$n^2 - 4 = 3^2 - 4 = 5$$

is prime.

6 ($\Rightarrow$) We prove this statement in the contrapositive. Suppose n is not even. Then $n = 2k + 1$ where $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k + 1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^4 + 6k^2 + 3k) + 1$$

is odd.

($\Leftarrow$) If n is even then $n = 2k$. Therefore,

$$n^3 = (2k)^3$$
$$= 8k^3$$
$$= 2(4k^3)$$

is even.

7 ($\Rightarrow$) Suppose that n is odd. Then $n = 2m + 1$, for some $m \in \mathbb{Z}$. Now either m is even or m is odd. If m is even, then $m = 2k$ so that

$$n = 2m + 1$$
$$= 2(2k) + 1$$
$$= 4k + 1.$$

as required. If m is odd then $m = 2q + 1$ so that

$$n = 2m + 1$$
$$= 2(2q + 1) + 1$$
$$= 4q + 3$$
$$= 4q + 4 - 1$$
$$= 4(q + 1) - 1$$
$$= 4k - 1, \text{ where } k = q + 1,$$

as required.
($\Leftarrow$) If $n = 4k \pm 1$ then either $n = 4k + 1$ or $n = 4k - 1$. If $n = 4k + 1$, then

$$n = 4k + 1$$
$$= 2(2k) + 1$$
$$= 2m + 1, \text{ where } m = 2k,$$

is odd, as required. Likewise, if

$n = 4k - 1$, then

$$\begin{aligned} n &= 4k - 1 \\ &= 4k - 2 + 1 \\ &= 2(2k - 1) + 1 \\ &= 2m + 1, \text{ where } m = 2k - 1, \end{aligned}$$

is odd, as required.

8 (⇒) Suppose that,

$$\begin{aligned} (x + y)^2 &= x^2 + y^2 \\ x^2 + 2xy + y^2 &= x^2 + y^2 \\ 2xy &= 0 \\ xy &= 0 \end{aligned}$$

Therefore, $x = 0$ or $y = 0$.
(⇐) Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\begin{aligned} (x + y)^2 &= (0 + y)^2 \\ &= y^2 \\ &= 0^2 + y^2 \\ &= x^2 + y^2, \end{aligned}$$

as required.

9 **a** Expanding gives

$$\begin{aligned} &(m - n)(m^2 + mn + n^2) \\ =&m^3 + m^2n + mn^2 - m^2n - mn^2 - n^3 \\ =&m^3 - n^3. \end{aligned}$$

b (⇐) We will prove this in the contrapositive. Suppose that $m - n$ were odd. Then either m is odd and n is even or visa versa.
Case 1 - If m is odd and n is even
The expression $m^2 + mn + n^2$ is of the form,

$$\text{odd} + \text{even} + \text{even} = \text{odd}.$$

Case 2 - m is even and n is odd
The expression $m^2 + mn + n^2$ is of the form,

$$\text{even} + \text{even} + \text{odd} = \text{odd}.$$

In both instances, the expression $m^2 + mn + n^2$ is odd. Therefore,

$$m^3 - n^3 = (m - n)(m^2 + mn + n^2)$$

is the product of two odd numbers, and will therefore be odd.

10 We first note that any integer n can be written in the form $n = 100x + y$ where $x, y \in \mathbb{Z}$ and y is the number formed by the last two digits. For example, $1234 = 100 \times 12 + 34$. Then

$$\begin{aligned} &n \text{ is divisible by } 4 \\ \Leftrightarrow &n = 100x + y = 4k, \text{ for some } k \in \mathbb{Z} \\ \Leftrightarrow &y = 4k - 100x \\ \Leftrightarrow &y = 4(k - 25x) \\ \Leftrightarrow &y \text{ is divisible by } 4. \end{aligned}$$

Solutions to Exercise 8E

1 a If we let $n = 31$ it is clear that

$$2n^2 - 4n + 31 = 2 \times 31^2 - 4 \times 31 + 31$$

is divisible by 31 and so cannot be prime.

b Let $x = 1$ and $y = -1$ so that

$$(x + y)^2 = (1 + (-1))^2 = 0,$$

while,

$$x^2 + y^2 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

c If $x = \frac{1}{2}$, then,

$$x^2 = \frac{1}{4} < \frac{1}{2} = x.$$

d If $n = 3$ then,

$$n^3 - n = 27 - 3 = 24$$

is even, although 3 is not.

e If $m = n = 1$ then $m + n = 2$ while $mn = 1$.

f Since 6 divides $2 \times 3 = 6$ but 6 does not divide 2 or 3, the statement is false.

2 a Negation: For all $n \in \mathbb{N}$, the number $9n^2 - 1$ is not a prime number.
Proof: Since

$$9n^2 - 1 = (3n - 1)(3n + 1),$$

and since each factor is greater than 1, the number $9n^2 - 1$ is not a prime number.

b Negation: For all $n \in \mathbb{N}$, the number $n^2 + 5n + 6$ is not a prime number.
Since

$$n^2 + 5n + 6 = (n + 2)(n + 3),$$

and since each factor is greater than 1, the number $9n^2 + 5n + 6$ is not a prime number.

c Negation: For all $x \in \mathbb{R}$, we have $2 + x^2 \neq 1 - x^2$
Proof: Suppose that $2 + x^2 = 1 - x^2$.
Rearranging the equation gives,

$$2 + x^2 = 1 - x^2$$
$$2x^2 = -1$$
$$x^2 = -\frac{1}{2},$$

which is impossible since $x^2 \geq 0$.

3 a Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of a and b are irrational, although $ab = 2$ is not.

b Let $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then clearly each of a and b are irrational, although $a + b = 0$ is not.

c Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of a and b are irrational, although $\frac{a}{b} = 1$ is not.

4 a If a is divisible by 4 then $a = 4k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 = (4k)^2 = 16k^2 = 4(4k^2)$$

is divisible by 4.

b Converse: If a^2 is divisible by 4 then a is divisible by 4.
This is clearly not true, since $2^2 = 4$ is divisible by 4, although 2 is not.

5 a If $a - b$ is divisible by 3 then

$a - b = 3k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 - b^2 = (a - b)(a + b) = 3k(a + b)$$

is divisible by 3.

b Converse: If $a^2 - b^2$ is divisible by 3 then $a - b$ is divisible by 3.
The converse is not true, since $2^2 - 1^2 = 3$ is divisible by 3, although $2 - 1 = 1$ is not.

6 a This statement is not true since for all $a, b \in \mathbb{R}$,

$$a^2 - 2ab + b^2 = (a - b)^2 \geq 0 > -1.$$

b This statement is not true since for all $x \in \mathbb{R}$, we have,

$$\begin{aligned} & x^2 - 4x + 5 \\ =& x^2 - 4x + 4 - 4 + 5 \\ =& (x - 2)^2 + 1 \\ \geq& 1 \\ >& \frac{3}{4}. \end{aligned}$$

7 a The numbers can be paired as follows:

$$\begin{aligned} 16 + 9 &= 25, & 15 + 10 &= 25 \\ 14 + 11 &= 25, & 13 + 12 &= 25 \\ 1 + 8 &= 9, & 2 + 7 &= 9, \\ 4 + 5 &= 9, & 3 + 6 &= 9. \end{aligned}$$

b We now list each number, in descending order, with each of its potential pairs.

12	4
11	5
10	6
9	7
8	1
7	2, 9
6	3, 10
5	4
4	5
3	1, 6
2	7
1	3, 8

Notice that the numbers 2 and 9 must be paired with 7. Therefore, one cannot pair all numbers in the required fashion.

8 If we let $x = c$, then

$$f(c) = ac^2 + bc + c = c(ac + b + 1)$$

is divisible by $c \geq 2$.

Solutions to Exercise 8F

1 a $\boxed{P(n)}$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1$$

and

$$\text{RHS} = \frac{1(1+1)}{2} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1 + 2 + \cdots + k + (k+1)$

$= \frac{k(k+1)}{2} + (k+1) \quad \text{(by (1))}$

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)((k+1)+1)}{2}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 + x$$

and

$$\text{RHS} = \frac{(1 - x^2)}{1 - x} = \frac{(1-x)(1+x)}{1-x} = 1 + x.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1 + x + x^2 + \cdots + x^k + x^{k+1}$

$= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \quad \text{(by (1))}$

$= \frac{1 - x^{k+1}}{1 - x} + \frac{x^{k+1}(1-x)}{1-x}$

$= \frac{1 - x^{k+1} + x^{k+1}(1-x)}{1-x}$

$= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1-x}$

$= \frac{1 - x^{k+2}}{1-x}$

$= \frac{1 - x^{(k+1)+1}}{1-x}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^2 - 1$$

and

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = 1.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \quad (1)$$

$P(k+1)$

LHS of $P(k+1)$

$$\begin{aligned}
&= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by (1))} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $P(n)$

$$1 \cdot 2 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

$P(1)$

If $n = 1$ then

$$\text{LHS} = 1 \times 2 = 2$$

and

$$\text{RHS} = \frac{1 \times 2 \times 3}{3} = 2.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$1 \cdot 2 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}. \quad (1)$$

$P(k+1)$

LHS of $P(k+1)$

$$\begin{aligned}
&= 1 \cdot 2 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by (1))} \\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\
&= \frac{(k+1)(k+2)(k+3)}{3} \\
&= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}
\end{aligned}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

e $P(n)$

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$P(1)$

If $n = 1$ then

$$\text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3}$$

and

$$\text{RHS} = \frac{1}{2 \times 1 + 1} = \frac{1}{3}.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}. \quad (1)$$

$P(k+1)$

LHS of $P(k+1)$

$$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \text{(by (1))}$$

$$=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{k+1}{(2k+3)}$$

$$=\frac{k+1}{(2(k+1)+1)}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $P(n)$

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

$P(2)$

If $n = 2$ then

$$\text{LHS} = 1-\frac{1}{2^2}=\frac{3}{4}$$

and

$$\text{RHS} = \frac{2+1}{2\times 2}=\frac{3}{4}.$$

Therefore $P(2)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

$P(k+1)$

LHS of $P(k+1)$

$$=\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \quad \text{(by (1))}$$

$$=\frac{k+1}{2k}\left(\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\frac{(k+1)(k^2+2k)}{2k(k+1)^2}$$

$$=\frac{k(k+1)(k+2)}{2k(k+1)^2}$$

$$=\frac{(k+2)}{2(k+1)}$$

$$=\frac{(k+1)+1}{2(k+1)}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a $P(n)$

$11^n - 1$ is divisible by 10

$P(1)$

If $n = 1$ then

$$11^1 - 1 = 11 - 1 = 10$$

is divisible by 10. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$11^k - 1 = 10m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} 11^{k+1} - 1 &= 11 \times 11^k - 1 \\ &= 11 \times (10m + 1) - 1 \quad \text{(by (1))} \\ &= 110m + 11 - 1 \\ &= 110m + 10 \\ &= 10(11m + 1) \end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $P(n)$

$3^{2n} + 7$ is divisible by 8

$P(1)$

If $n = 1$ then

$$3^{2\times1} + 7 = 9 + 7 = 16 = 2 \times 8$$

is divisible by 8. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$3^{2k} + 7 = 8m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} 3^{2(k+1)} + 7 &= 3^{2k+2} + 7 \\ &= 3^{2k} \times 3^2 + 7 \\ &= (8m - 7) \times 9 + 7 \quad \text{(by (1))} \\ &= 72m - 63 + 7 \\ &= 72m - 56 \\ &= 8(9m - 7) \end{aligned}$$

is divisible by 8. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $P(n)$

$7^n - 3^n$ is divisible by 4

$P(1)$

If $n = 1$ then

$$7^1 - 3^1 = 7 - 3 = 4$$

is divisible by 4. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$7^k - 3^k = 4m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$P(k+1)$

$$\begin{aligned} &7^{k+1} - 3^{k+1} \\ &= 7 \times 7^k - 3^{k+1} \\ &= 7 \times (4m + 3^k) - 3 \times 3^k \quad \text{(by (1))} \\ &= 28m + 7 \times 3^k - 3 \times 3^k \\ &= 28m + 4 \times 3^k \\ &= 4(7m + 3^k) \end{aligned}$$

is divisible by 4. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$5^n + 6 \times 7^n + 1$ is divisible by 4

$\boxed{P(1)}$

If $n = 1$ then

$$5^1 + 6 \times 7^1 + 1 = 48 = 4 \times 12$$

is divisible by 4. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$5^k + 6 \times 7^k + 1 = 4m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$5^{k+1} + 6 \times 7^{k+1} + 1$

$=5 \times 5^k + 6 \times 7 \times 7^k + 1$

$=5 \times (4m - 6 \times 7^k - 1) + 42 \times 7^{k+1}$

$=20m - 30 \times 7^k - 5 + 42 \times 7^k + 1$

$=20m + 12 \times 7^k - 4$

$=4(5m + 3 \times 7^k - 1)$

is divisible by 4. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

3 a $\boxed{P(n)}$

$4^n > 10 \times 2^n$ where $n \geq 4$

$\boxed{P(4)}$

If $n = 4$ then

LHS $= 4^4 = 256$ and RHS $= 10 \times 2^4 = 160$.

Since LHS > RHS, $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^k > 10 \times 2^k \text{ where } k \geq 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$4^{k+1} > 10 \times 2^{k+1}.$$

LHS of $P(k + 1) = 4^{k+1}$

$= 4 \times 4^k$

$> 4 \times 10 \times 2^k$ (by (1))

$= 40 \times 2^k$ (as 10 > 2)

$= 20 \times 2^{k+1}$

$> 10 \times 2^{k+1}$

$=$ RHS of $P(k + 1)$

Therefore $P(k + 1)$ is true.

Since $P(5)$ is true and $P(k + 1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$3^n > 5 \times 2^n$ where $n \geq 5$

$\boxed{P(5)}$

If $n = 5$ then

LHS $= 3^5 = 243$ and RHS $= 5 \times 2^5 = 160$.

Since LHS > RHS, $P(5)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k > 5 \times 2^k \text{ where } k \geq 5. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$3^{k+1} > 5 \times 2^{k+1}.$$

LHS of $P(k + 1) = 3^{k+1}$

$= 3 \times 3^k$

$> 3 \times 5 \times 2^k$ (by (1))

$= 15 \times 2^k$ (as 10 > 2)

$> 10 \times 2^k$

$= 5 \times 2^{k+1}$

$=$ RHS of $P(k + 1)$

Therefore $P(k + 1)$ is true.

Since $P(5)$ is true and $P(k + 1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 5$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$2^n > 2n$ where $n \geq 3$

$\boxed{P(3)}$

If $n = 3$ then

$$\text{LHS} = 2^3 = 8 \text{ and RHS} = 2 \times 3 = 6.$$

Since LHS > RHS, $P(3)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$2^k > 2k \text{ where } k \geq 3. \qquad (1)$$

$\boxed{P(k + 1)}$

We have to show that

$$2^{k+1} > 2(k + 1).$$

$$\begin{aligned}
\text{LHS of } P(k + 1) &= 2^{k+1} \\
&= 2 \times 2^k \\
&> 2 \times 2k \quad \text{(by (1))} \\
&= 4k \\
&= 2k + 2k \\
&\geq 2k + 2 \quad (\text{as } 2k \geq 2) \\
&= 2(k + 1) \\
&= \text{RHS of } P(k + 1)
\end{aligned}$$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all integers $n \geq 3$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$n! > 2^n$ where $n \geq 4$

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4! = 24 \text{ and RHS} = 2^4 = 16.$$

Since LHS > RHS, $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k! > 2^k \text{ where } k \geq 4. \qquad (1)$$

$\boxed{P(k + 1)}$

We have to show that

$$(k + 1)! > 2^{k+1}.$$

$$\begin{aligned}
\text{LHS of } P(k + 1) &= (k + 1)! \\
&= (k + 1)k! \\
&> (k + 1) \times 2^k \quad \text{(by (1))} \\
&> 2 \times 2^k \quad (\text{as } k + 1 > 2) \\
&= 2^{k+1} \\
&= \text{RHS of } P(k + 1)
\end{aligned}$$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

4 a $\boxed{P(n)}$

$a_n = 2^n + 1$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = a_1 = 3 \text{ and RHS} = 2^1 + 1 = 3.$$

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 2^k + 1. \qquad (1)$$

$\boxed{P(k + 1)}$

We have to show that

$$a^{k+1} = 2^{k+1} + 1.$$

$$\begin{aligned}
\text{LHS of } P(k+1) &= a_{k+1} \\
&= 2a_k - 1 \quad \text{(by definition)} \\
&= 2(2^k + 1) - 1 \quad \text{(by (1))} \\
&= 2^{k+1} + 2 - 1 \\
&= 2^{k+1} + 1 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $P(n)$

$a_n = 5^n - 1$

$P(1)$

If $n = 1$ then

LHS = a_1 = 4 and RHS = $5^1 - 1 = 4$.

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$a_k = 5^k - 4. \qquad (1)$$

$P(k+1)$

We have to show that

$$a^{k+1} = 5^{k+1} - 4.$$

$$\begin{aligned}
\text{LHS} &= a_{k+1} \\
&= 5a_k + 4 \quad \text{(by definition)} \\
&= 5(5^k - 1) + 4 \quad \text{(by (1))} \\
&= 5^{k+1} - 5 + 4 \\
&= 5^{k+1} - 1 \\
&= \text{RHS}
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $P(n)$

$a_n = 2^n + n$

$P(1)$

If $n = 1$ then

LHS = a_1 = 3 and RHS = $2^1 + 1 = 3$.

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$a_k = 2^k + k. \qquad (1)$$

$P(k+1)$

We have to show that

$$a^{k+1} = 2^{k+1} + k + 1.$$

$$\begin{aligned}
\text{LHS of } P(k+1) &= a_{k+1} \\
&= 2a_k - k + 1 \quad \text{(by definition)} \\
&= 2(2^k + k) - k + 1 \quad \text{(by (1))} \\
&= 2^{k+1} + 2k - k + 1 \\
&= 2^{k+1} + k + 1 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

5 $P(n)$

3^n is odd where $n \in \mathbb{N}$

$P(1)$

If $n = 1$ then clearly

$$3^1 = 3$$

is odd. Therefore, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$3^k = 2m + 1 \qquad (1)$$

for some $m \in \mathbb{Z}$.

$P(k+1)$

$$3^{k+1} = 3 \times 3^k$$
$$= 3 \times (2m + 1) \quad \text{(by (1))}$$
$$= 6m + 3$$
$$= 6m + 2 + 1$$
$$= 2(3m + 1) + 1$$

is odd, so that $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 a $\boxed{P(n)}$

$n^2 - n$ is even, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^2 - \times 1 = 0$$

is even. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^2 - k$ is even. Therefore,

$$k^2 - k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k + 1)}$

$(k + 1)^2 - (k + 1)$

$= k^2 + 2k + 1 - k - 1$

$= k^2 + k$

$= (k^2 - k) + 2k$

$= 2m + 2k \quad$ (by (1))

$= 2(m + k)$

Since this is even, $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^2 - n = n(n - 1).$$

As this is the product of two consecutive numbers, one of them must be even, so that the product will also be even.

7 a $\boxed{P(n)}$

$n^3 - n$ is divisible by 3, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^3 - 1 = 0$$

is divisible by 3. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^3 - k$ is divisible by 3. Therefore,

$$k^3 - k = 3m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k + 1)}$

We have to show that $(k+1)^3 - (k+1)$ is divisible by 3.

$(k + 1)^3 - (k + 1)$

$= k^3 + 3k^2 + 3k + 1 - k - 1$

$= k^3 - k + 3k^2 + 3k$

$= (k^3 - k) + 3k^2 + 3k$

$= 3m + 3k^2 + 3k \quad$ (by (1))

$= 3(m + k^2 + k)$

Since this is divisible by 3, $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^3 + n = n(n^2 - 1) = n(n - 1)(n + 1).$$

As this is the product of three consecutive numbers, one of them must be divisible by 3, so that the product will also be divisible by 3.

8 **a**

n	1	2	3	4	5
a_n	9	99	999	9999	99999

b We claim that $a_n = 10^n - 1$.

c $P(n)$

$a_n = 10^n - 1$

$P(1)$

If $n = 1$, then

LHS $= a_1 = 9$ and RHS $= 10^1 - 1 = 9$.

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$a_k = 10^k - 1. \qquad (1)$$

$P(k+1)$

We have to show that

$$a^{k+1} = 10^{k+1} - 1.$$

$$\begin{aligned}\text{LHS} &= a_{k+1}\\ &= 10a_k + 9 \quad \text{(by definition)}\\ &= 10(10^k - 1) + 9 \quad \text{(by (1))}\\ &= 10^{k+1} - 10 + 9\\ &= 10^{k+1} - 1\\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

9 **a**

n	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	3	5	8	13	21	34	55

b $P(n)$

$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

$P(1)$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_3 - 1 = 2 - 1 = 1.$$

Since LHS = RHS, $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$f_1 + f_2 + \cdots + f_k = f_{k+2} - 1. \qquad (1)$$

$P(k+1)$

$$\begin{aligned}\text{LHS of } P(k+1) &= f_1 + f_2 + \cdots + f_k + f_{k+1}\\ &= f_{k+2} - 1 + f_{k+1} \quad \text{(by (1))}\\ &= f_{k+1} + f_{k+2} - 1\\ &= f_{k+3} - 1 \quad \text{(by definition)}\\ &= f_{(k+1)+2} - 1\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $f_1 = 1$

$f_1 + f_3 = 1 + 2 = 3$

$f_1 + f_3 + f_5 = 3 + 5 = 8$

$f_1 + f_3 + f_5 + f_7 = 8 + 13 = 21$

d From the pattern observed above, we claim that

$$f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}.$$

e $P(n)$

$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

$P(1)$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_2 - 1 = 2 - 1 = 1.$$

Since LHS = RHS, $P(1)$ is true.

| $P(k)$ |

Assume that $P(k)$ is true so that

$$f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}. \quad (1)$$

| $P(k+1)$ |

$$\begin{aligned}\text{LHS} &= f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1}\\ &= f_{2k} + f_{2k+1} \quad \text{(by (1))}\\ &= f_{2k+2} \quad \text{(by definition)}\\ &= f_{2(k+1)}\\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f | $P(n)$ |

The Fibonacci number f_{3n} is even.

| $P(1)$ |

If $n = 1$ then

$$f_3 = 2$$

is even, therefore $P(1)$ is true.

| $P(k)$ |

Assume that $P(k)$ is true so that f_{3k} is even. That is,

$$f_{3k} = 2m \quad (1)$$

for some $m \in \mathbb{Z}$.

| $P(k+1)$ |

$$\begin{aligned}f_{3(k+1)} &= f_{3k+3}\\ &= f_{3k+2} + f_{3k+1} \quad \text{(by definition)}\\ &= f_{3k+1} + f_{3k} + f_{3k+1}\\ &= 2f_{3k+1} + f_{3k}\\ &= 2f_{3k+1} + 2m \quad \text{(by (1))}\\ &= 2(f_{3k+1} + m)\end{aligned}$$

Since this is even, $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 | $P(n)$ |

Since we're only interested in odd numbers our proposition is:
$4^{2n-1} + 5^{2n-1}$ is divisible by 9, where $n \in \mathbb{N}$.

| $P(1)$ |

If $n = 1$ then

$$4^1 + 5^1 = 9$$

is divisible by 9. Therefore $P(1)$ is true.

| $P(k)$ |

Assume that $P(k)$ is true so that

$$4^{2k-1} + 5^{2k-1} = 9m \qquad (1)$$

for some $k \in \mathbb{Z}$.

| $P(k+1)$ |

The next odd number will be $2k + 1$. Therefore, we have to prove that

$$4^{2k+1} + 5^{2k+1}$$

is divisible by 9.

$4^{2k+1}+5^{2k+1}$

$=4^2\times 4^{2k-1}+5^2\times 5^{2k-1}$

$=16\times(9m-5^{2k-1})+25\times 5^{2k-1}$ (by (1))

$=144m-16\times 5^{2k-1}+25\times 5^{2k-1}$

$=144m+9\times 5^{2k-1}$

$=9(16+5^{2k-1})$

Since this is divisible by 9, we've shown that $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n\in\mathbb{N}$ by the principle of mathematical induction.

11 $P(n)$

A set of numbers S with n numbers has a largest element.

$P(1)$

If $n=1$, then set S has just one element. This single element is clearly the largest element in the set.

$P(k)$

Assume that $P(k)$ is true. This means that a set of numbers S with k numbers has a largest element.

$P(k+1)$

Suppose set S has $k+1$ numbers. Remove one of the elements, say x, so that we now have a set with k numbers. The reduced set has a largest element, y. Put x back in set S, so that its largest element will be the larger of x and y. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n\in\mathbb{N}$ by the principle of mathematical induction.

12 $P(n)$

It is possible to walk around a circle whose circumference includes n friends and n enemies (in any order) without going into debt.

$P(1)$

If $n=1$, there is one friend and one enemy on the circumference of a circle. Start your journey at the friend, receive \$1, then walk around to the enemy and lose \$1. At no point will you be in debt, so $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true. This means that it is possible to walk around a circle with k friends and k enemies (in any order) without going into debt, provided you start at the correct point.

$P(k+1)$

Suppose there are $k+1$ friends and $k+1$ enemies located on the circumference of the circle, in any order. Select a friend whose next neighbour is an enemy (going clockwise), and remove these two people. As there are now k friends and k enemies, it is possible to walk around the circle without going into debt, provided you start at the correct point. Now reintroduce the two people, and start walking from the same point. For every part of the journey you'll have the same amount of money as before except when you meet the added friend, who gives you \$1, which is immediately lost to the added enemy.

Therefore $P(n)$ is true for all $n\in\mathbb{N}$ by the principle of mathematical induction.

13 $P(n)$

Every integer j such that $2\le j\le n$ is divisible by some prime.

$P(2)$

If $n = 2$, then $j = 2$ is clearly divisible by a prime, namely itself. Therefore $P(2)$ is true.

$P(k)$

Assume that $P(k)$ is true. Therefore, every integer j such that $2 \leq j \leq k$ is divisible by some prime.

$P(k+1)$

We need to show that integer j such that $2 \leq j \leq k+1$ is divisible by some prime. By the induction assumption, we already know that every j with $2 \leq j \leq k$ is divisible by some prime. We need only prove that $k + 1$ is divisible by a prime. If $k + 1$ is a prime number, then we are finished. Otherwise we can find integers a and b such that $k + 1 = ab$ and $2 \leq a \leq k$ and $2 \leq b \leq k$. By the induction assumption, the number a will be divisible by some prime number. Therefore $k + 1$ is divisible by some prime number.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14 If such a colouring of the regions is possible we will call it a **satisfactory colouring**.

$P(n)$

If n lines are drawn then the resulting regions have a satisfactory colouring.

$P(1)$

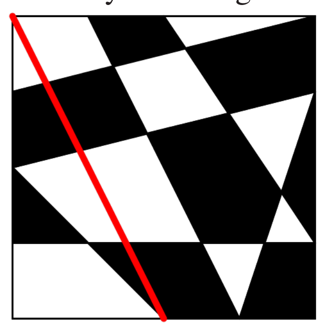
If $n = 1$, then there is just one line. We colour one side black and one side white. This is a satisfactory colouring. Therefore $P(1)$ is true.

$P(k)$

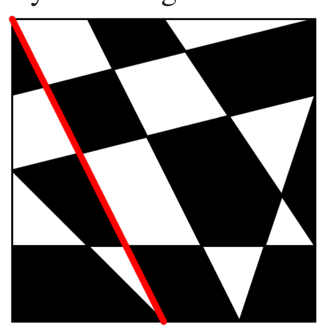
Assume that $P(k)$ is true. This means that we can obtain a satisfactory colouring if there are k lines drawn.

$P(k+1)$

Now suppose that there are $k + 1$ lines drawn. Select one of the lines, and remove it. There are now k lines, and the resulting regions have a satisfactory colouring since we assumed $P(k)$ is true. Now add the removed line. This will divide some regions into into two new regions with the same colour, so this is not a satisfactory colouring.



However, if we switch each colour on **one** side of the line we obtain a satisfactory colouring.



This is because inverting a satisfactory colouring will always give a satisfactory colouring, and regions separated the new line will not have the same colour.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to Technology-free questions

1 a Let the 3 consecutive integers be $n, n+1$ and $n+2$. Then,

$$n + (n+1) + (n+2) = 3n + 3$$
$$= 3(n+1)$$

is divisible by 3.

b This statement is not true. For example, $1 + 2 + 3 + 4 = 10$ is not divisible by 4

2 (Method 1) If n is even then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$n^2 - 3n + 1 = (2k)^2 - 2(2k) + 1$$
$$= 4k^2 - 4k + 1$$
$$= 2(2k^2 - 2k) + 1$$

is odd.

(Method 2) If n is even then $n^2 - 3n + 1$ is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

3 a (Contrapositive) If n is not even, then n^3 is not even. (Alternative) If n is odd, then n^3 is odd.

b If n is odd then $n = 2k + 1$, for some $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k+1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^3 + 6k^2 + 3k) + 1$$

is odd.

c This will be a proof by contradiction. Suppose $\sqrt[3]{6}$ is rational so that $\sqrt[3]{6} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^3 = 6q^3 \qquad (1)$$
$$\Rightarrow \quad p^3 \text{ is divisible by 2}$$
$$\Rightarrow \quad p \text{ is divisible by 2}$$
$$\Rightarrow \quad p = 2k \text{ for some } k \in \mathbb{N}$$
$$\Rightarrow \quad (2k)^3 = 6q^3 \text{(substitituting into (1))}$$
$$\Rightarrow \quad 8k^3 = 6q^3$$
$$\Rightarrow \quad 4k^2 = 3q^2$$
$$\Rightarrow \quad q^2 \text{ is divisible by 2}$$
$$\Rightarrow \quad q \text{ is divisible by 2.}$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

4 a Suppose n is the first of three consecutive numbers. If n is divisible by 3 then there is nothing to prove. Otherwise, it is of the form $n = 3k + 1$ or $n = 3k + 2$. In the first case,

$$n = 3k + 1$$
$$n + 1 = 3k + 2$$
$$n + 2 = 3k + 3 = 3(k+1)$$

so that the third number is divisible by 3. In the second case,

$$n = 3k + 2$$
$$n + 1 = 3k + 3 = 3(k+1)$$
$$n + 2 = 3k + 4$$

so that the second number is divisible by 3.

b The expression can be readily factorised so that

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2)$$
$$= n(n + 1)(n + 2)$$

is the product of 3 consecutive integers. As one of these integers must be divisible by 3, the product must also be divisible by 3.

5 a if m and n are divisible by d then $m = pd$ and $n = qd$ for some $p, q \in \mathbb{Z}$. Therefore,

$$m - n = pd - qd$$
$$= d(p - q)$$

is divisible by d.

b Take any two consecutive numbers n and $n + 1$. If d divides n and $n + 1$ then d must divide $(n + 1) - n = 1$. As the only integer that divides 1 is 1, the highest common factor must be 1, as requried.

c We know that any factor of 1002 and 999 must also divide $1002 - 999 = 3$. As the only factors of 3 are 1 and 3, the highest common factor must be 3.

6 a If $x = 9$ and $y = 16$ then the left hand side equals

$$\sqrt{9 + 16} = \sqrt{25} = 5$$

while the right hand side equals

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

b (⇒)

$$[t]\ \sqrt{x + y} = \sqrt{x} + \sqrt{y}$$
$$\Rightarrow x + y = \left(\sqrt{x} + \sqrt{y}\right)^2$$
$$\Rightarrow x + y = x + \sqrt{xy} + y$$
$$\Rightarrow 0 = sqrtxy$$
$$\Rightarrow xy = 0$$
$$\Rightarrow x = 0 \text{ or } y = 0$$

(⇐) Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\sqrt{x + y} = \sqrt{y + 0}$$
$$= \sqrt{y}$$
$$= \sqrt{y} + \sqrt{0}$$
$$= \sqrt{y} + \sqrt{x},$$

as required.

7 (Case 1) If n is even then the expression is of the form

even + even + even = even.

(Case 1) If n is odd then the expression is of the form

odd + odd + even = even.

8 a If $a = b = c = d = 1$ then the left hand side equals

$$\frac{1}{1} + \frac{1}{1} = 2$$

while the right hand side equals

$$\frac{1 + 1}{1 + 1} = 1.$$

b first note that if $\dfrac{c}{d} > \dfrac{a}{b}$ then $bc > ad$.

Therefore,

$$\frac{a+c}{b+d} - \frac{a}{b}$$
$$= \frac{b(a+c)}{b(b+d)} - \frac{a(b+d)}{b(b+d)}$$
$$= \frac{b(a+c) - a(b+d)}{b(b+d)}$$
$$= \frac{ab+bc-ab-ad}{b(b+d)}$$
$$= \frac{bc-ad}{b(b+d)}$$
$$> 0$$

since $bc > ad$. This implies that

$$\frac{a+c}{b+d} > \frac{a}{b}.$$

Similarly, we can show that

$$\frac{a+c}{b+d} < \frac{c}{d}.$$

9 a $\boxed{P(n)}$

$6^n + 4$ is divisible by 10

$\boxed{P(1)}$

If $n = 1$ then

$$6^1 + 4 = 10$$

is divisible by 10. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$6^k + 4 = 10m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned} 6^{k+1} + 4 &= 6 \times 6^k + 4 \\ &= 6 \times (10m - 4) + 4 \quad \text{(by (1))} \\ &= 60m - 24 + 4 \\ &= 60m - 20 \times 3^k \\ &= 10(6m - 2) \end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$\boxed{P(1)}$

If $n = 1$ then LHS$= 1^2 = 1$and

$$\text{RHS} = \frac{1(2 \times 1 - 1)(2 \times 1 + 1)}{3} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^2 + 3^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$= 1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2$$
$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \text{(by (1))}$$
$$= \frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3}$$
$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$
$$= \frac{(2k+1)(k(2k-1) + 3(2k+1))}{3}$$
$$= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$$
$$= \frac{(2k+1)(2k+3)(k+1)}{3}$$
$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$
$$= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$=$RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to multiple-choice questions

1 E The expression $m - 3n$ is of the form

$$\text{even} - \text{odd} = \text{odd}.$$

2 E If m is divisible by 6 and n is divisible by 15 then $m = 6p$ and $n = 15q$ for $p, q \in \mathbb{Z}$. Therefore,

$$m \times n = 90pq$$

$$m + n = 6p + 15q = 3(2p + 5q)$$

From these two expressions, it should be clear that A,B,C and D are true, while E might be false. For example, if $m = 6$ and $n = 15$ then $m + n = 21$ is not divisible by 15.

3 C We obtain the contrapositive by switching P and Q and negating both. Therefore, the contrapositive will be

$$\text{not } Q \Rightarrow \text{not } P$$

.

4 B We obtain the converse by switching P and Q. Therefore, the converse will be

$$Q \Rightarrow P$$

.

5 C If $m + n = mn$ then

$$n = mn - m$$

$$n = m(n - 1)$$

This means that n is divisible by $n - 1$, which is only possible if $n = 2$ or $n = 0$. If $n = 0$, then $m = 0$. If $n = 2$, then $m = 2$. Therefore there are only two solutions, $(0, 0)$ and $(2, 2)$.

6 D The only statement that is true for all real numbers a, b and c is D. Counterexamples can be found for each of the other expressions, as shown below.

A $\quad \frac{1}{3} < \frac{1}{2}$

B $\quad \frac{1}{2} > \frac{1}{-1}$

C $\quad 3 \times -1 < 2 \times -1$

E $\quad 1^2 < (-2)^2$

7 D As n is the product of 3 consecutive integers, one of which will be divisible by 3 and one of which will be divisible by 2. The product will be then be divisible by 1, 2, 3 and 6. On the other hand, it won't necessarily be divisible by 5 since $2 \times 3 \times 4$ is not divisible by 5.

8 C Each of the statements is true except the third. In this instance, $1 + 3$ is even, although 1 and 3 are not even.

Solutions to extended-response questions

1 **a** The number of dots can be calculated two ways, either by addition,

$$(1 + 2 + 3 + 4) + (1 + 2 + 3 + 4)$$

or by multiplication,

$$4 \times 5.$$

Equating these two expressions gives,

$$\begin{aligned}(1 + 2 + 3 + 4) + (1 + 2 + 3 + 4) &= 4 \times 5\\ 2(1 + 2 + 3 + 4) &= 4 \times 5\\ 1 + 2 + 3 + 4 &= \frac{4 \times 5}{2}\end{aligned}$$

The argument obviously generalises to more dots, giving equation (1).

b We have,

$$\begin{aligned}1 + 2 \cdots + 99 &= \frac{99 \times 100}{2}\\ &= 99 \times 50,\end{aligned}$$

which is divisible by 99.

c Suppose that m is the first number, so that the n connective numbers are

$$m, m + 1, \ldots, m + n - 1.$$

Then,

$$\begin{aligned}&m + (m + 1) + (m + 2) + \cdots + (m + n - 1)\\ &= n \times m + (1 + 2 + \cdots (n - 1))\\ &= nm + \frac{(n - 1)n}{2}\\ &= n\left(m + \frac{n - 1}{2}\right)\end{aligned}$$

Since n is odd, $n - 1$ is even. This means that $\frac{n - 1}{2}$ is an integer. Therefore, the term in brackets is an integer, which means the expression is divisible by n.

d Since

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2},$$

we need to prove the following statement:

$P(n)$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$$

P(1)

If $n = 1$ then

$$\text{LHS} = 1^3 = 1$$

and

$$\text{RHS} = \frac{1^2(1+1)^2}{4} = 1.$$

Therefore $P(1)$ is true.

P(k)

Assume that $P(k)$ is true so that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}. \quad (1)$$

P(k + 1)

$$\begin{aligned}
&\text{LHS of } P(k+1) \\
&= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{(by (1))} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a The first number is divisible by 2, the second by 3, the third by 4 and so on. As each number has a factor greater than 1, each is a composite number. Therefore this is a sequence of 9 consecutive composite numbers.

b We consider the this sequence of 10 consecutive numbers,

$$11! + 2, 11! + 3, \ldots, 11! + 11.$$

The first number is divisible by 2, the second by 3 and so on. Therefore as each number has a factor greater than 1, each is a composite number.

3 **a** Since (a, b, c) is a Pythagorean triple, we know that $a^2 + b^2 = c^2$. Then (na, nb, nc) is also a Pythagorean triple since,

$$\begin{aligned}(na)^2 + (nb)^2 &= n^2a^2 + n^2b^2\\ &= n^2(a^2 + b^2)\\ &= n^2(c^2)\\ &= (nc)^2,\end{aligned}$$

as required.

b Suppose that $(n, n + 1, n + 2)$ is a Pythagorean triple. Then

$$\begin{aligned}n^2 + (n + 1)^2 &= (n + 2)^2\\ n^2 + n^2 + 2n + 1 &= n^2 + 4n + 4\\ n^2 - 2n - 3 &= 0\\ (n - 3)(n + 1) &= 0\\ n &= 3, -1.\end{aligned}$$

However, since $n > 0$, we obtain only one solution, $n = 3$, which corresponds to the famous $(3, 4, 5)$ triangle.

c Suppose some triple (a, b, c) contained the number 1. Then clearly, 1 will be the smallest number. Therefore, we can suppose that

$$\begin{aligned}1^2 + b^2 &= c^2\\ c^2 - b^2 &= 1\\ (c - b)(c + b) &= 1\end{aligned}$$

Since the only divisor of 1 is 1, we must have

$$\begin{aligned}&c + b = 1\\ &c - b = 1\\ &\Rightarrow b = 0 \text{ and } c = 1.\end{aligned}$$

This is a contradiction, since b must be a positive integer. Now suppose some triple (a, b, c) contained the number 2. Then 2 will be smallest number. Therefore, we can suppose that

$$\begin{aligned} 2^2 + b^2 &= c^2 \\ c^2 - b^2 &= 4 \\ (c-b)(c+b) &= 4 \end{aligned}$$

Since the only divisors of 4 are 1, 2 and 4, we must have

$$c + b = 4$$
$$c - b = 1$$
$$\Rightarrow b = \frac{3}{2}, c = \frac{5}{2}$$

or

$$c + b = 2$$
$$c - b = 2$$
$$\Rightarrow b = 0, c = 2$$

In both instances, we have a contradiction since b must be a positive integer.

4 a (Case 1) If $a = 3k + 1$ then

$$\begin{aligned} a^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

leaves a remainder of 1 when divided by 3.
(Case 2) If $a = 3k + 2$ then

$$\begin{aligned} a^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

also leaves a remainder of 1 when divided by 3.

b Suppose by way of contradiction that neither a nor b are divisible by 3. Then using the previous question, each of a^2 and b^2 leave a remainder of 1 when divided by 3. Therefore $a^2 = 3k + 1$ and $b^2 = 3m + 1$, for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 3k + 1 + 3m + 1 \\ &= 3(k + m) + 2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 3, which is not possible.

5 **a** $\boxed{P(n)}$

$n^2 + n$ is divisible by 2, where $n \in \mathbb{Z}$.

$\boxed{P(1)}$

If $n = 1$ then $1^2 + 1 = 2$ is divisible by 2. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k^2 + k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

Letting $n = k + 1$ we have,

$$
\begin{aligned}
&(k+1)^2 + (k+1) \\
=&k^2 + 2k + 1 + k + 1 \\
=&k^2 + 3k + 2 \\
=&(k^2 + k) + (2k + 2) \\
=&2m + 2(k+1) \qquad \text{(by (1))} \\
=&2(m + k + 1)
\end{aligned}
$$

is divisible by 2. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Since

$$n^2 + n = n(n+1)$$

is the product of two consecutive integers, one of them must be even. Therefore the product will also be even.

c If n is odd, then $n = 2k + 1$. Therefore

$$
\begin{aligned}
n^2 - 1 &= (2k+1)^2 - 1 \\
&= 4k^2 + 4k + 1 - 1 \\
&= 4k^2 + 4k \\
&= 4k(k+1) \\
&= 4 \times 2k \qquad \text{(since the product of consecutive integers is even)} \\
&= 8k
\end{aligned}
$$

as required.

6 **a** If n is divisible by 8, then $n = 8k$ for some $k \in \mathbb{Z}$. Therefore

$$n^2 = (8k)^2 = 64k^2 = 8(8k^2)$$

is divisible by 8.

b (Converse) If n^2 is divisible by 8, then n is divisible by 8.

c The converse is not true. For example, $4^2 = 16$ is divisible by 8 however 4 is not divisible by 8.

7 **a** There are many possibilities. For example $3 + 97 = 100$ and $5 + 97 = 102$.

b Suppose 101 could be written as the sum of two prime numbers. Then one of these primes must be 2, since all other pairs of primes have an even sum. Therefore $101 = 2 + 99$, however 99 is not prime.

c There are many possibilities. For example, $7 + 11 + 83 = 101$.

d Consider any odd integer n greater than 5. Then $n - 3$ will be an even number greater than 2. If the Goldbach Conjecture is true, then $n - 3$ is the sum of two primes, say p and q. Then $n = 3 + p + q$, as required.

8 **a** We have,

$$\begin{aligned}\frac{1}{n-1} - \frac{1}{n} &= \frac{n}{n(n-1)} - \frac{n-1}{n(n-1)}\\ &= \frac{n-(n-1)}{n(n-1)}\\ &= \frac{n-n+1}{n(n-1)}\\ &= \frac{1}{n(n-1)}.\end{aligned}$$

b Using the identity developed in the previous question, we have,

$$\begin{aligned}&\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{n(n+1)}\\ =&\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n}\\ =&\frac{1}{1} - \frac{1}{n}\\ =&1 - \frac{1}{n}\end{aligned}$$

as required.

c

d Since $k^2 > k(k-1)$ for all $k \in \mathbb{N}$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2}$$

$$= \frac{1}{1^2} + \left(\frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2}\right)$$

$$< \frac{1}{1^2} + \left(\frac{1}{2 \times 1} + \frac{1}{3 \times 2} \cdots + \frac{1}{n(n-1)}\right)$$

$$= \frac{1}{1^2} + 1 - \frac{1}{n}$$

$$= 2 - \frac{1}{n}$$

$$< 2,$$

as required.

9 a We have,

$$\frac{x+y}{2} - \sqrt{xy} = \frac{a^2+b^2}{2} - \sqrt{a^2b^2}$$

$$= \frac{a^2+b^2}{2} - ab$$

$$= \frac{a^2+b^2}{2} - \frac{2ab}{2}$$

$$= \frac{a^2-2ab+b^2}{2}$$

$$= \frac{(a-b)^2}{2}$$

$$\geq 0.$$

It is also worth noting that we get equality if and only if $x = y$.

b **i** Using the above inequality, we obtain,

$$a + \frac{1}{a} \geq 2\sqrt{a \cdot \frac{1}{a}}$$

$$= 2\sqrt{1}$$

$$= 2.$$

as required.

ii Using the above inequality three times, we obtain,

$$\begin{aligned}(a+b)(b+c)(c+a) &\geq 2\sqrt{ab} \times 2\sqrt{bc} \times 2\sqrt{ca} \\ &= 8(\sqrt{a})^2(\sqrt{b})^2(\sqrt{c})^2 \\ &= 8abc,\end{aligned}$$

as required.

iii This inequality is a little trickier. We have,

$$\begin{aligned}a^2+b^2+c^2 &= \left(\frac{a^2}{2}+\frac{b^2}{2}\right)+\left(\frac{b^2}{2}+\frac{c^2}{2}\right)+\left(\frac{a^2}{2}+\frac{c^2}{2}\right) \\ &= \frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{a^2+c^2}{2} \\ &\geq \sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{a^2c^2} \\ &= ab+bc+ac,\end{aligned}$$

as required.

c If a rectangle has length x and width y then its perimeter will be $2x + 2y$. A square with the same perimeter will have side length,

$$\frac{2x+2y}{4} = \frac{x+y}{2}.$$

Therefore,

$$A(\text{square}) = \left(\frac{x+y}{2}\right)^2 \geq xy = A(\text{rectangle}).$$

10 We show that it is only possible for Kaye to be the liar.

case 1

Suppose Jaye is lying

⇒ Kaye is not lying

⇒ Elle is lying

⇒ There are two liars

⇒ This is impossible.

case 2

Suppose Kaye is lying

⇒ Jaye is not lying and Elle is not lying

⇒ Kaye is the only liar

case 3

Suppose Elle is lying

⇒ Mina is not lying

⇒ Karl is lying

⇒ There are two liars

⇒ This is impossible.

11 First note that the four sentences can be recast as:

- Exactly three of these statements are true.
- Exactly two of these statements are true.
- Exactly one of these statements are true.
- None of these statements are true.

At most one of these statements can be true, or else we obtain a contradiction. If none of the statements is true, then the last statement is true. This means that at least one of the statements is true. This also gives a contradiction. Therefore, only one of the statements is true, that is, the third statement.

12 **a** There is only one possibility,

1, 2, 4, 8	3, 5, 6, 7

b We know that we can split the numbers $1, 2, \dots, 8$,

1, 2, 4, 8	3, 5, 6, 7

Deleting the largest number, 8, will give a splitting of $1, 2, \dots, 7$.

1, 2, 4	3, 5, 6, 7

Continuing this process, deleting the 7, will be a splitting of the numbers $1, 2, \dots, 6$, and so on.

c We first note that if a set can be split then two numbers can't appear in the same group as their difference. To see this, if x and y and $x - y$ all belong to the same group then $(x - y) + y = x$. Let's now try to split the numbers $1, 2, \dots, 9$. Call the two groups X and Y. We can assume that $1 \in X$. We now consider four cases for the groups containing elements 2 and 9.

(case 1) Suppose $2 \in X$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
(assumed)	9		
		3	$(1, 2 \in X)$
		7	$(2, 9 \in X)$
$(3, 7 \in Y)$	4		
		5	$(1, 4 \in X)$
		6	$(2, 4 \in X)$
$(5, 6 \in Y)$	8		

This doesn't work, since X is forced to contain the numbers $1, 8$ and 9.

(case 2) Suppose $2 \in X$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
		9	(assumed)
		3	$(1, 2 \in X)$
$(3, 9 \in Y)$	6		
		4	$(2, 6 \in X)$
		5	$(1, 6 \in X)$

This doesn't work, since Y is forced to contain the numbers $4, 5$ and 9.

(case 3) Suppose $2 \in Y$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
(assumed)	9		
		8	$(1, 9 \in X)$
$(2, 8 \in Y)$	6		
		3	$(6, 8 \in X)$
$(2, 8 \in Y)$	5		$(3, 8 \in X)$

This doesn't work, since X is forced to contain the numbers $1, 5$ and 6.

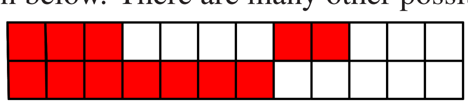
(case 4) Suppose $2 \in Y$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
		9	(assumed)
$(2, 9 \in Y)$	7		
		6	$(1, 7 \in X)$
$(2, 8 \in Y)$	4		
		3	$(4, 7 \in X)$

This doesn't work, since Y is forced to contain the numbers $3, 6$ and 9.

d If the numbers $1, 2, \ldots, n$ could be split, where $n \geq 9$, then we could successively eliminate the largest term to obtain a splitting of the numbers $1, 2, \ldots, 9$. However, we already know that this is impossible.

13 a A suitable tiling is shown below. There are many other possibilities.



b Tile E must go into a corner. This is because there are only two other tiles (A and

B) that it can go next to. Tile F must also go into a corner. This is because there are only two other tiles (B and C) that it can go next to.

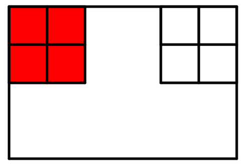
(Case 1) Tile E and tile F are in different rows

Since tile B must go next to both tiles E and F, this is impossible.

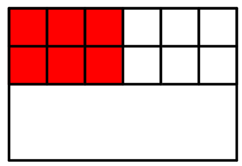
(Case 2) Tile E and tile F are in the same row

Assume tile F is in the top left position.

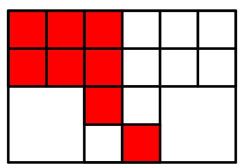
Then tile E goes in the top right position.



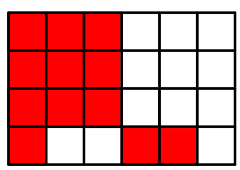
Therefore tile B must go between them.



Tile C must then go beneath tile F and tile A must go beneath tile E. Consequently, tile D must go beneath tile B. Therefore, there is only one valid orientation of tile D.



This fixes the orientation of tiles A and C.



Since tile F could have gone into any one of the four corners, there are only four ways to tile the grid.